AN ELEMENTARY APPROACH TO NON-SYMMETRIC UPPER BOUNDS FOR POWER MEANS

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In the field of mathematical inequalities, the study of means has always been a central theme. One of the most significant concepts is the power mean of order k (also known as the generalized mean), defined for a set of positive real numbers x_1, x_2, …, x_n as:

$$M_k(x_1,\dots,x_n)=\left(\frac{1}{n}\sum_{i=1}^{n}x_i^k\right)^{\frac{1}{k}}.$$

The behavior of M_k is well-understood through classical results such as the Power Mean Inequality, which states that M_k is a non-decreasing function of k [1]. While symmetric inequalities like the Arithmetic Mean-Geometric Mean (AM-GM) inequality and Jensen's inequality provide elegant bounds for these means [2], they often prove insufficient when dealing with «tight» problems or cases where the variables are significantly distant from each other. In competitive mathematics and advanced analysis, there arises a need for sharper, more adaptable bounds [3].

This article explores a specific approach often associated with Mildorf's Lemma. Unlike traditional bounds that maintain symmetry between variables, the method presented here focuses on establishing a non-symmetric upper bound [4]. The primary goal of this paper is to derive these bounds using strictly elementary methods, avoiding the complexities of higher-order calculus or Taylor expansions. By doing so, we make the technique accessible to students and researchers alike. Furthermore, we will demonstrate the practical power of this lemma by applying it to several challenging problems that are otherwise difficult to settle using standard symmetric tools [5].

Let us start our journey with a known result.

Lemma 1. If a and b are real numbers such that $a\ge b\ge 0$ and k is a positive integer, then for all $c_k\in\left(0,\frac{1}{\sqrt[k]{2}-1}\right]$ the following inequality is true:

$$\sqrt[k]{a^k+b^k}\le a+\frac{b}{c_k}.$$

Proof. We have

$$\sqrt[k]{a^k+b^k}\le a+\frac{b}{c_k}\ \Leftrightarrow\ a^k+b^k\le a^k+\sum_{i=1}^{k}C_k^i\cdot a^{k-i}\cdot\frac{b^i}{c_k^i}.$$

From $a\ge b$ it follows that

$$a^k+\sum_{i=1}^{k}C_k^i\cdot a^{k-i}\cdot\frac{b^i}{c_k^i}\ge a^k+\sum_{i=1}^{k}C_k^i\cdot\frac{b^k}{c_k^i}=a^k+b^k\cdot\left[\sum_{i=1}^{k}C_k^i\cdot\left(\frac{1}{c_k}\right)^i\right]=$$

$$=a^k+b^k\cdot\left[\left(1+\frac{1}{c_k}\right)^k-1\right]\ge a^k+b^k\ \Leftrightarrow$$

$$\Leftrightarrow\ \left(1+\frac{1}{c_k}\right)^k-1\ge 1\Leftrightarrow 1+\frac{1}{c_k}\ge\sqrt[k]{2}\Leftrightarrow c_k\le\frac{1}{\sqrt[k]{2}-1}.$$

It is not difficult to see that $c_k=\frac{1}{\sqrt[k]{2}-1}$ is the best constant, because, for $a=b\ne 0$, $a\sqrt[k]{2}\le a+\frac{a}{c_k}\Rightarrow\sqrt[k]{2}\le 1+\frac{1}{c_k}\Rightarrow c_k\le\frac{1}{\sqrt[k]{2}-1}$. Equality occurs when $a=b$ or $b=0$. □

Lemma 2. If $a_0\ge a_1\ge a_2\ge\ldots\ge a_n$ are positive real numbers, then the following inequality is satisfied for all $c_k\in\left(0,\frac{1}{\sqrt[k]{2}-1}\right]$:

$$\sqrt[k]{a_0^k+a_1^k+a_2^k+\ldots+a_n^k}\le a_0+\frac{a_1}{c_k}+\frac{a_2}{c_k^2}+\ldots+\frac{a_n}{c_k^n}.$$

Equality holds if and only if $a_i=\sqrt[k]{2^{n-i-1}}\cdot m$, $i=\overline{0,n-1}$, and $m=a_n$.

Proof. Applying Lemma 1,

$$\sqrt[k]{a_0^k+a_1^k+\ldots+a_n^k}=\sqrt[k]{a_0^k+(a_1^k+\ldots+a_n^k)}\le a_0+\frac{\sqrt[k]{a_1^k+\ldots+a_n^k}}{c_k},$$

$$\sqrt[k]{a_1^k+a_2^k+\ldots+a_n^k}=\sqrt[k]{a_1^k+(a_2^k+\ldots+a_n^k)}\le a_1+\frac{\sqrt[k]{a_2^k+\ldots+a_n^k}}{c_k},$$

$$\ldots$$

$$\sqrt[k]{a_{n-2}^k+a_{n-1}^k+a_n^k}=\sqrt[k]{a_{n-2}^k+(a_{n-1}^k+a_n^k)}\le a_{n-2}+\frac{\sqrt[k]{a_{n-1}^k+a_n^k}}{c_k},$$

$$\sqrt[k]{a_{n-1}^k+a_n^k}\le a_{n-1}+\frac{a_n}{c_k}.$$

Combining these n inequalities we get the desired result. Equality occurs if and only if $a_i=\sqrt[k]{a_{i+1}^k+a_{i+2}^k+\ldots+a_n^k}$, $i=\overline{0,n-1}$, i.e. $a_i=\sqrt[k]{2^{n-i-1}}\cdot m$, $i=\overline{0,n-1}$. So, the main result that we proved in this article is:

$$\sqrt[k]{a_0^k+a_1^k+\ldots+a_n^k}\le a_0+a_1(\sqrt[k]{2}-1)+a_2(\sqrt[k]{2}-1)^2+\ldots+a_n(\sqrt[k]{2}-1)^n,$$

for all $a_0\ge a_1\ge a_2\ge\ldots\ge a_n>0$. □

Next, we shall examine practical examples regarding the application of the aforementioned lemmas.

Example 1. Let a and b two non-negative real numbers with $a\ge b$. Prove that the following inequality holds:

$$\sqrt{a^2+b^2}+\sqrt[3]{a^3+b^3}+\sqrt[4]{a^4+b^4}\le 3a+b.$$

Solution. From Lemma 1,

$$\sqrt{a^2+b^2}\le a+b(\sqrt{2}-1),$$
$$\sqrt[3]{a^3+b^3}\le a+b(\sqrt[3]{2}-1),$$
$$\sqrt[4]{a^4+b^4}\le a+b(\sqrt[4]{2}-1).$$

By adding up these inequalities we obtain

$$\sqrt{a^2+b^2}+\sqrt[3]{a^3+b^3}+\sqrt[4]{a^4+b^4}\le 3a+b(\sqrt{2}+\sqrt[3]{2}+\sqrt[4]{2}-3)\le$$
$$\le 3a+0,9b\le 3a+b.$$

Equality occurs if and only if $b=0$. As an observation, this problem cannot be solved by Power Mean inequality or Mildorf's Lemma. Also, it is not sufficient to observe that $\sqrt[k]{a^k+b^k}\le a+\frac{b}{k}$, for $k=2,3,4$, because $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\approx 1{,}083>1$.

Example 2. Let a, b, c be the side lengths of a triangle, with $a+b+c=1$, and let $n\ge 2$ be an integer. Prove that

$$\sqrt[n]{a^n+b^n}+\sqrt[n]{b^n+c^n}+\sqrt[n]{c^n+a^n}<1+\frac{\sqrt[n]{2}}{2}.$$

Solution. Without loss of generality, suppose that $a\ge b\ge c$. As a, b, c are the side lengths of a triangle,

$$b+c>a\Rightarrow 1-a>a\Rightarrow a<\frac{1}{2} \tag{1}$$

Now, using Lemma 1 we obtain

$$\sqrt[n]{a^n+b^n}\le a+b(\sqrt[n]{2}-1),$$
$$\sqrt[n]{b^n+c^n}\le b+c(\sqrt[n]{2}-1),$$
$$\sqrt[n]{c^n+a^n}\le a+c(\sqrt[n]{2}-1).$$

Adding up,

$$\sum_{cyc}\sqrt[n]{a^n+b^n}\le 2a+b\sqrt[n]{2}+2c(\sqrt[n]{2}-1)=(a+b+c)+a+(b+2c)(\sqrt[n]{2}-1)-c=$$
$$=1+a+(b+2c)(\sqrt[n]{2}-1)-c<1+a+(b+c)(\sqrt[n]{2}-1)=1+a+(1-a)(\sqrt[n]{2}-1)=$$
$$=a(2-\sqrt[n]{2})+\sqrt[n]{2}<1+\frac{\sqrt[n]{2}}{2}\Leftrightarrow a(2-\sqrt[n]{2})<\frac{2-\sqrt[n]{2}}{2}\Leftrightarrow a<\frac{1}{2},$$

which is clearly true, from (1).

In this article, the problem of determining non-symmetric upper bounds for the power mean of order k of n numbers was thoroughly investigated using elementary mathematical methods. The use of elementary methods instead of complex mathematical analysis opens up new methodological possibilities for evaluating fundamental concepts such as power means. In particular, Mildorf's Lemma and the two primary lemmas derived from it served to express non-symmetric upper bounds in a precise and accessible manner. The lemmas presented and their subsequent proofs maintain a balance between mathematical rigor and simplicity. This approach can serve as a methodological guide for teaching the theory of power means and breaking down complex inequalities into simpler components.

The application of these theoretical results through two concrete examples demonstrated the effectiveness of this method in solving high-level mathematical competition problems. The skill of applying non-symmetric bounds provides students and researchers with a strategic advantage when tackling complex algebraic challenges. In summary, the proposed elementary approach not only enriches the theory of power means but also provides a unique algorithm for problem-solving that can be adapted to other types of means in the future.

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Darajali o‘rta qiymatlar uchun simmetrik bo‘lmagan yuqori chegaralarni topishda elementar yondashuv

Annotatsiya. Ushbu maqolaning maqsadi faqat elementar usullardan foydalangan holda n ta sonning k-tartibli darajali o‘rta qiymati uchun yuqori chegarani (maqolada ko‘rib o‘tilganidek, simmetrik bo‘lmagan) topish va undan ba’zi tatbiqlarda qanday foydalanishni ko‘rsatishdir.

Kalit so‘zlar: Mildorf lemmasi, simmetrik bo‘lmagan yuqori chegara, elementar usullar, matematik musobaqalar.

Элементарный подход к нахождению несимметричных верхних границ для степенных средних

Аннотация. Целью данной статьи является нахождение верхней границы (которая, как будет показано, не является симметричной) для степенного среднего порядка k от n чисел с использованием только элементарных методов, а также рассмотрение способов её применения в некоторых прикладных задачах.

Ключевые слова: лемма Мильдорфа, несимметричная верхняя граница, элементарные методы, математические соревнования.

An elementary approach to non-symmetric upper bounds for power means

Abstract. The purpose of this article is to find an upper bound (but not symmetric, as we will see) for the power mean of order k of n numbers using just elementary methods, and to see how to use it in some applications.

Key words: Mildorf’s lemma, non-symmetric upper bound, elementary methods, mathematical competitions.
